

# Frozen Temporal Pattern in Growing Systems

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## Introduction

The coloured patterns of seashells demonstrate great variety. Fascinated by the complexity of the patterns, we were astonished at the similarities between then and the patterns we obtained when trying to model the heterogeneously catalysed oxidation of carbon monoxide.

H. Meinhardt has modelled many shell patterns using a set of coupled one-dimensional differential equations [1–3], whereas our cellular automaton model for the catalysis is based on a discrete description of the process of pattern formation. We have coupled linearly a large set of reactors, characterized by their state  $x(t)$  and their phase  $p(t)$ . The coincidence of our results with the shell patterns encouraged us to apply our model to the growth patterns of seashells as well.

Shells grow at their borders [2, 4], and so Meinhardt worked with one-dimensional differential equations. Instead of using a one-dimensional continuum as he does, we take a one-dimensional cellular automation [5–8], the cells of which now represent biological cells in the border of the shells. The crucial point is that we assume the creation of a cell in the new front at time  $(t + 1)$  to depend on the current situation of its mother cell in the actual front at

time  $t$  and the situation of the cells neighbouring the mother cell at time  $t$ . The temporal sequence of states of the one-dimensional automaton thus represents the development of the moving front in the growing shells. The pattern of the shell is nothing other than the frozen set of all temporal states of the front which have been created during the growth of the system.

## The Model

Let us assume that the incorporation of pigments in a new cell  $i$  (daughter cell) at time  $(t + 1)$  which is born from its mother cell by cell division depends upon the concentration  $x(i, t)$  of the reactants (pre-pigments) in its mother and in both of her neighbouring cells  $(i - 1)$  and  $(i + 1)$ , which are “aunt” cells to the daughter cells.

Moreover, it is reasonable to assume that each cell  $i$  can exist in at least two different states of activity  $p(i, t) \in \{0, 1\}$  at time  $t$ . The activity of the mother cell influences the amount of prepigment  $x(i, t + 1)$  which the daughter cell inherits from her mother. On the other hand, the daughter’s activity  $p(i, t + 1)$  at time  $(t + 1)$  is determined by the concentration of the prepigments of her mother  $x(i, t)$  and her aunts  $x(i - 1, t)$  and  $x(i + 1, t)$  as well as by the activity  $p(i, t)$  of her mother.

Therefore, the situation  $z(i, t)$  of a cell  $i$  at time  $t$  is characterized by a vector

$$z(i, t) = (i, t, p, x) = \begin{pmatrix} x(i, t) \\ p(i, t) \end{pmatrix} \quad (1)$$

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We can formulate a transformation rule  $T$ , which determines the situation  $z(i, t + 1)$  of the daughter cell  $i$  at time  $(t + 1)$ :

$$T: z(i, t) \rightarrow z(i, t + 1)$$

$$z(i, t + 1) = \begin{pmatrix} x(i, t + 1) \\ p(i, t + 1) \end{pmatrix} = \begin{pmatrix} f\{p(i, t), x(i, t) + x(i - 1, t) + x(i + 1, t)\} \\ g\{p(i, t), x(i, t) + x(i - 1, t) + x(i + 1, t)\} \end{pmatrix} \quad (2)$$

$f$  and  $g$  are discrete functions which can be represented by  $2 \times k$  matrices:

$$\begin{pmatrix} f(0, 0) & f(0, 1) & f(0, 2) & \dots & f(0, k) \\ f(1, 0) & f(1, 1) & f(1, 2) & \dots & f(1, k) \end{pmatrix} \quad (3)$$

with  $f(p, m) \in X$ ;  $m = 0, 1, 2, \dots, k$ ;  $p \in \{0, 1\}$ , where  $X$  is the set of the possible numbers (concentration) of the prepigments  $X = \{0, 1, 2, 3, \dots\}$  and  $k$  is the largest number of prepigments which can be reached via the addition of the number of prepigments of the cell  $i$  and its neighbouring cells  $(i - 1)$  and  $(i + 1)$  at time  $t$ . The function  $g$  is given by:

$$\begin{pmatrix} g(0, 0) & g(0, 1) & g(0, 2) & \dots & g(0, k) \\ g(1, 0) & g(1, 1) & g(1, 2) & \dots & g(1, k) \end{pmatrix} \quad (4)$$

with  $g(p, m) \in P$ , where  $P$  is the set of possible activities of a cell:  $P = \{0, 1\}$ . Table 1 shows an example for a rule which will be used later on.

This transformation  $T$  (Eq. 2) is performed at the same time  $t$  for all cells of the automaton.

To obtain a temporal pattern one has to introduce some special cells into the starting automaton at time  $t = 0$  whose situation differs from that of all others. At least one cell  $j$  should have a small number of prepigments  $x(j, 0) > 0$ , while the number of prepigments should be zero in all the other cells. However, all cells  $i$  of the automaton may have the same high activity  $p(i, 0) = 0$ .

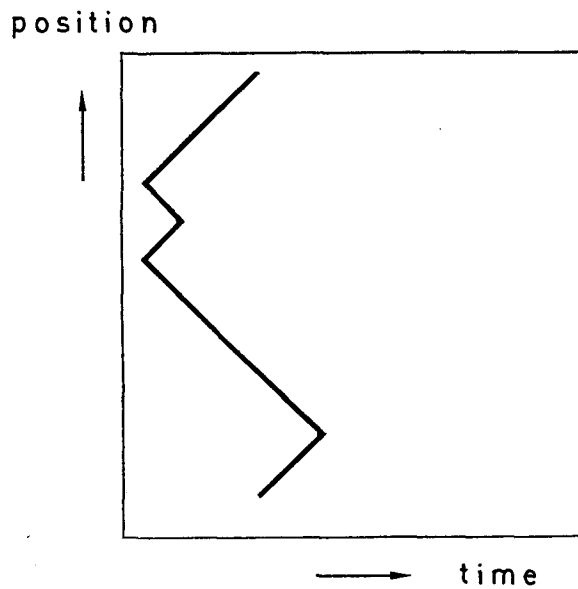
For colouring, the prepigments have to be transformed into the pigments. This process may depend upon the activity and the number of prepigments in the cells. Even the same patterns can be coloured differently, accentuating amounts of special prepigments or the activity of the cells, or only a different way of transforming the situation of a cell into its pigment colour.

In this way, a large variety of coloured patterns can be produced, some of which may resemble the observed seashell patterns.

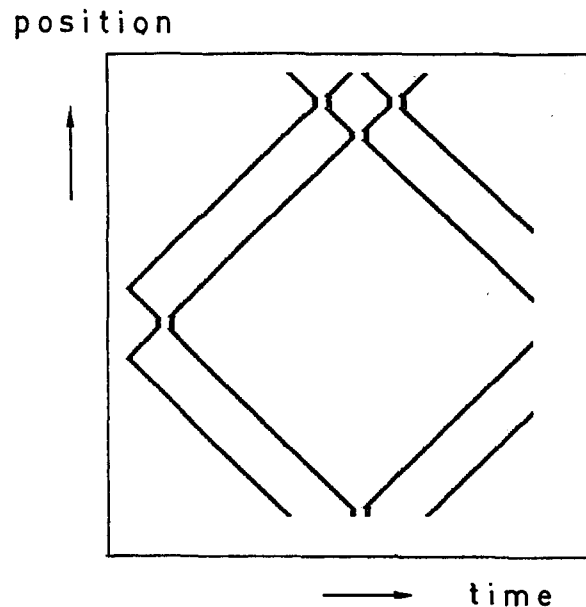
**Table 1.** Vector automaton: transformation rule (Eq. 2), 1/6-8/5 rule

| The function $f(p(i, t), x_s(i, t))$ |               | $x_s(i, t)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12    | 13 ... |
|--------------------------------------|---------------|-------------|---|---|---|---|---|---|---|---|---|---|----|----|-------|--------|
| $p(i, t) = 0$                        | $x(i, t + 1)$ | 0           | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4  | 5  | 5 ... |        |
| $p(i, t) = 1$                        | $x(i, t + 1)$ | 0           | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 2 | 2 | 2  | 3  | 3 ... |        |
| The function $g(p(i, t), x_s(i, t))$ |               | $x_s(i, t)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12    | 13 ... |
| $p(i, t) = 0$                        | $p(i, t + 1)$ | 0           | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1  | 1  | 1 ... |        |
| $p(i, t) = 1$                        | $p(i, t + 1)$ | 0           | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1  | 1  | 1 ... |        |

$x_s(i, t) = x(i - 1, t) + x(i, t) + x(i + 1, t)$ ; this special rule is called the 1/6-8/5 rule, where 1 and 6 are the number of zero values in the first and second row of the  $f$  matrix respectively, and 8 and 5 mean the corresponding number of zero values in the  $g$  matrix.



**Fig. 1.** The activity waves which travel along the one-dimensional automaton (1/6-8/4 rule) like a chemical wave. The automaton is circularly closed. Two special points in the starting automaton (*left side*) have been used

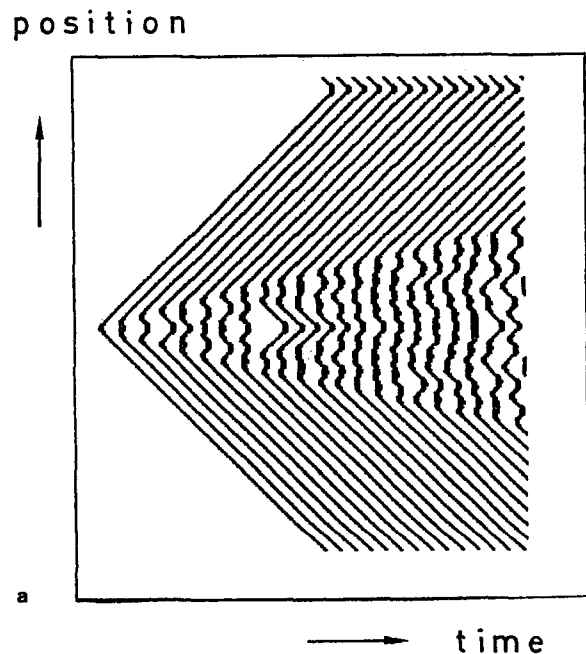


**Fig. 2.** The activity waves which travel along the one-dimensional automaton (1/6-6/5 rule) like solitons. The automaton is circularly closed. Two starting points (*left side*) have been used

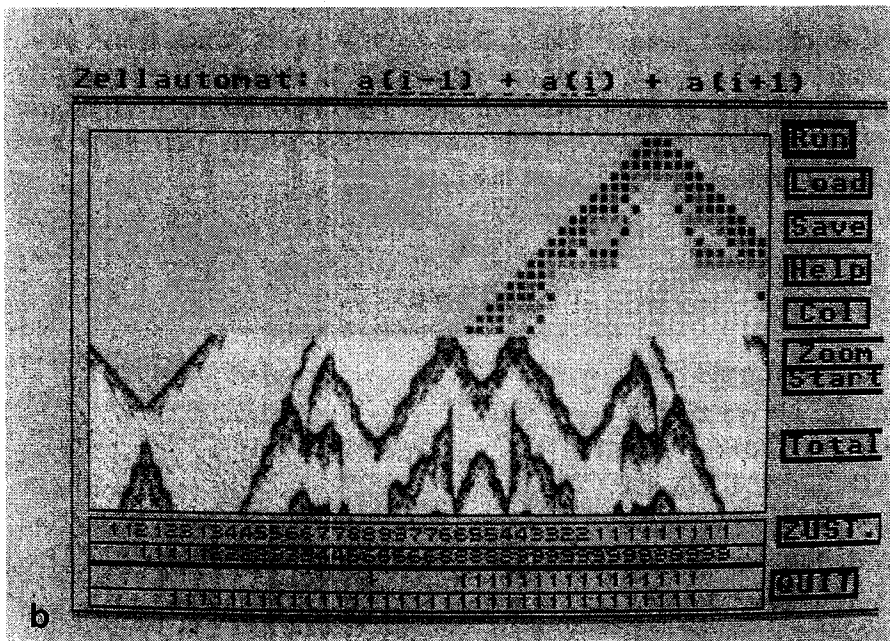
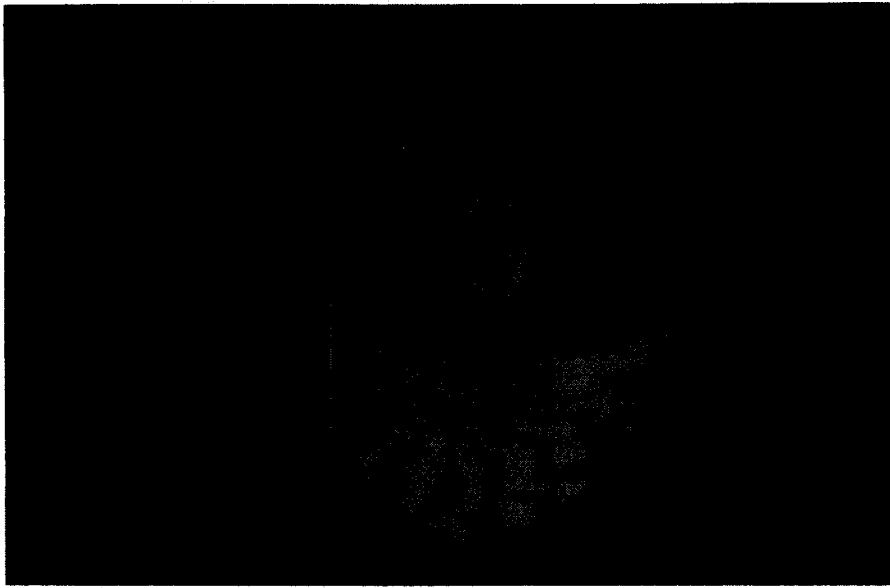
### Resulting Patterns

The possible patterns of the simple one-dimensional automata, the states of the cells of which are scalars, have been classified by Wolfram [5]. Apart from his well-known fractal patterns, the one-dimensional vector automaton model presented here exhibits new patterns like chemical waves (also named autowaves) [9, 10] and solitons [11, 12].

Such waves travel along the one-dimensional automaton and, if they behave like chemical waves, may annihilate each other (Fig. 1). They also may cross each other with a phase shift if they are travelling like solitons (Fig. 2). But there are also various waves which exhibit quite different behaviour. For instance, one can observe oscillators periodically creating waves which travel along the



**Fig. 3. a** An automaton with an oscillator (1/6-8/5 rule) which produces a sequence of wave fronts, between which a fractal core is created. **b** A feather of a bird exhibiting a pattern which resembles the pattern of the fractal core in a



**Fig. 4. a** The pigmentation of a top shell of the family Trochidae. **b** Coloured pattern of the number of prepigments produced by the one-

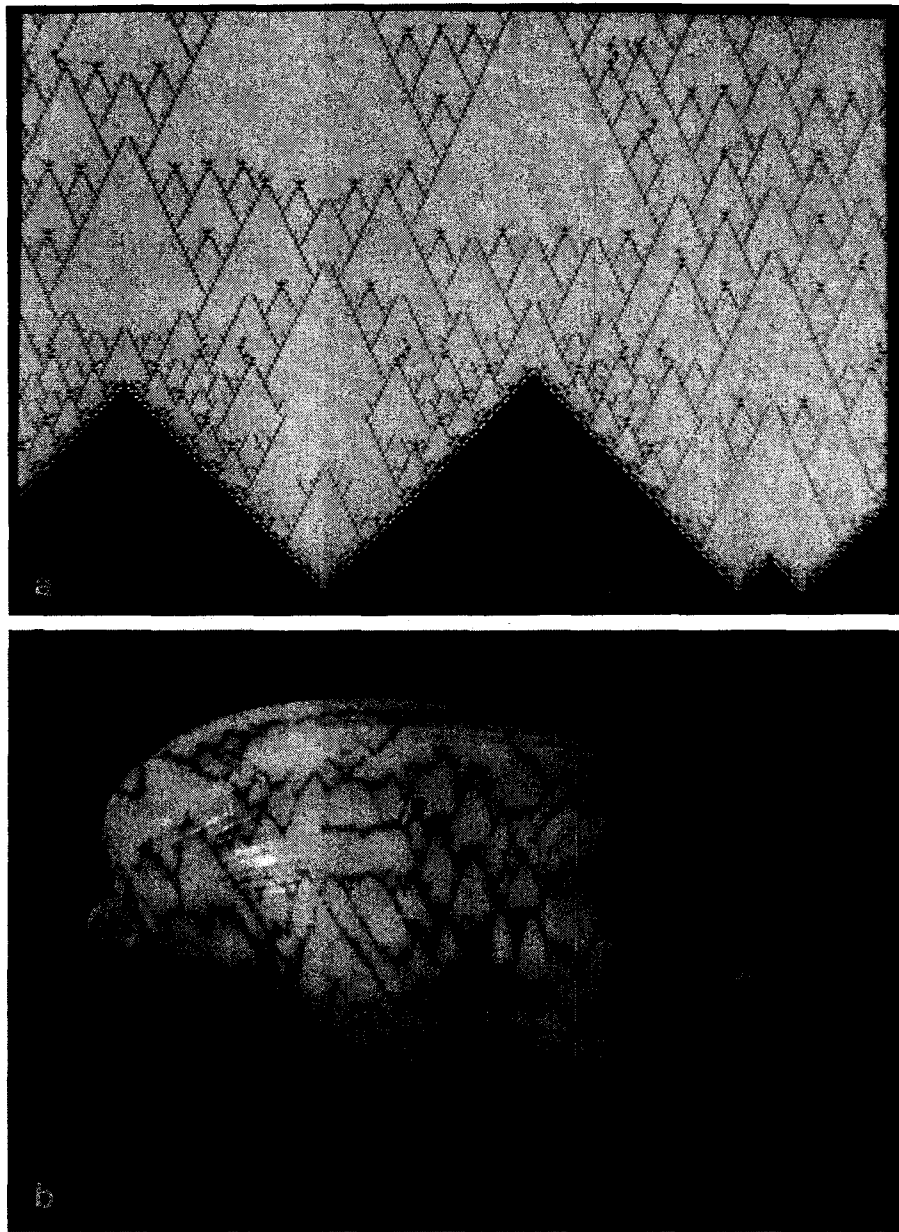
dimensional vector automaton. The rule is shown at the *bottom*

automaton showing interference with each other.

There is another very interesting behaviour (see Fig. 3 a): a wave consisting of two dispersing one-dimensional excited states periodically creates an excited block in between. The end points of the blocks become the starting points of two new waves, which will behave like the original wave. By this means, the wave is creating a fractal core inside itself. Figure 3 b shows the pattern of a feather which greatly resembles this fractal wave.

Starting not just with one special cell but with a few of them, which may be distributed randomly, a universe of different and complex patterns can be produced. Among these patterns there are classes which strongly resemble the frozen pattern of the seashells (see Fig. 4).

What is really astonishing is the major role of the fractals among the pattern in the seashells (see Fig. 5). For example, the *Cymbiolacca* shell exhibits a pattern of brown triangles of different sizes. The way the triangles are interlocked is typical for patterns of penetrating fractal Sier-



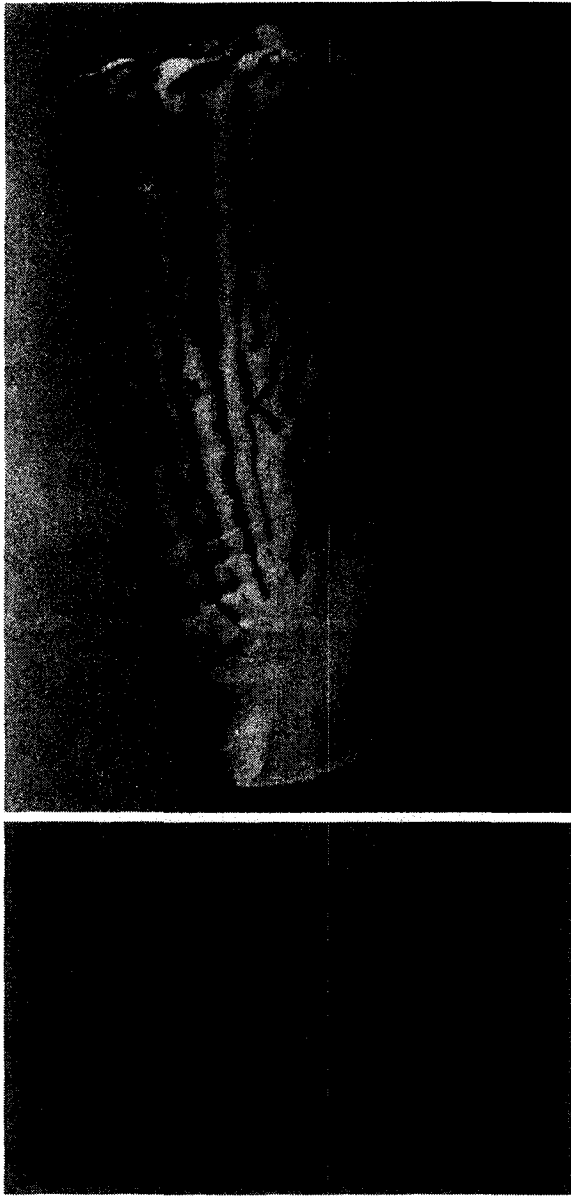
**Fig. 5. a** A *Cymbiolacca* shell (family Volutidae) with a fractal pigment pattern. **b** An

automaton, the pattern of which resembles the main elements of *Cymbiolacca* pigmentation

pinsky gaskets [13]. Sometimes one can observe showers of small triangles, while on other positions the sides of large triangles run through pale yellow-grey parts of the shell. A book of seashell patterns look like a zoo of fractal patterns and their combinations, which can be observed especially on cone shells [14, 15]. Another typical fractal pattern can be seen in the shell of *Conus princeps* (Fig. 6). Simulating this pattern with our automaton machine, it can be classified by the type of interpenetrating core waves which have been mentioned above.

### Concluding Remarks

The regularities and the irregularities in the pattern of the shells can be reproduced by the automaton model if fractal patterns interact starting from different positions at time  $t = 0$ . What is so fascinating about the fractal patterns? Fractals are strongly related to the occurrence of deterministic chaos [16], which does not mean that one loses all regularities but only the simple symmetries such as translational or rotational symmetry. If all these symmetries break down in a



**Fig. 6. a** *Conus princeps* (Linnaeus 1758), Sinaloa, Mexico. **b** The pattern of an automaton which resembles the main elements of the pigmentation of the *Conus princeps*

state of chaos, one very characteristic symmetry survives: the dilatation symmetry, which is mostly disregarded. In the pattern of the seashells it is precisely this dilatation symmetry which plays the major role, since all the other symmetries vanished.

The growth mechanism of the pattern proposed above is based on a strictly ordered one-dimensional arrangement of cells. This is sensible in the case of the seashells but one can also develop the method for two- or three-dimensional

processes to explain other phenomena in pure chemical or living systems. Even reaction in fluids can be modelled in such a way, if the cells stay together for a suitable period of time [17–19]. It would be of great interest to look for the spreading of the patterns of excited cells, even in flowing systems.

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## References

1. Meinhardt H (1984) Models for positional signaling, the threefold subdivision of segments and the pigmentation patterns of molluscs. *J Embryol Exp Morphol [Suppl]* 83:289–311
2. Meinhardt H, Klingler M (1987) A model for pattern formation on the shells of molluscs. *J Theor Biol* 126:63–89
3. Gierer A, Meinhardt H (1972) A theory of biological pattern formation. *Kybernetik* 12:30–39
4. Meinhardt H (1987) Bildung geordneter Strukturen bei der Entwicklung höherer Organismen. In: Küppers B-O (ed) *Ordnung aus dem Chaos*. Piper, Munich, pp 215–241
5. Wolfram S (1984) Universality and complexity in cellular automata. *Physica* 10D:1–35
6. Toffoli T, Margolus N (1987) *Cellular automata machines – a new environment for modelling*. MIT Press, Cambridge
7. Plath PJ (1989) *Modelling of heterogeneously catalyzed reactions by cellular automata of dimension between one and two. Optimal structures in heterogeneous reaction systems*. Springer, Berlin Heidelberg New York (Springer series in synergetics, vol 44)
8. Schwietering J, Plath PJ. *Wachsende Muster*. Submitted to *Wissenschaft und Fortschritt*, Akademie-Verlag, Berlin (submitted for publication)
9. Kuramoto Y (1984) *Chemical oscillations, waves, and turbulence*. Springer, Berlin Heidelberg New York (Springer series in synergetics, vol 19)

10. Krinsky VI (1984) Self-organization – autowaves and structures far from equilibrium. Springer, Berlin Heidelberg New York (Springer series in synergetics, vol 28)
11. Rebbi C (1979) Solitonen. Spektrum Wiss 4:63–78
12. Eilenberger G (1983) Solitons – mathematical methods for physicists. Springer, Berlin Heidelberg New York (Springer series in solid-state sciences, vol 19)
13. Mandelbrodt BB (1987) Die Fraktale Geometrie der Natur. Akademie, Berlin
14. Walls JG (1978) Cone shells – a synopsis of the living conidae. TFH, Hong Kong
15. Wilson BR, Gillett K (1971) Australian shells. Tuttle, Rutland
16. Peitgen H-O, Richter PH (1986) The beauty of fractals – images of complex dynamical systems. Springer, Berlin Heidelberg New York
17. Gerhardt M (1987) Mathematische Modellierung der Dynamik der heterogen katalysierten Oxidation von Kohlenmonoxid: Numerische Behandlung eines diskreten mathematischen Modells von über Diffusion miteinander gekoppelter Speicher. Thesis, University of Bremen
18. Gerhardt M, Schuster H (1989) A cellular automaton describing the formation of spatially ordered structures in chemical systems. Physica D 36:209–221
19. Dewdney AK (1989) Wellen aus der Computer Retorte. Spektrum Wiss [Sonderh] 8:38–41